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Fractional Fokker-Planck equation for Lévy flights in nonhomogeneous environments

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The fractional Fokker-Planck equation, which contains a variable diffusion coefficient, is discussed and solved. It corresponds to the Lévy flights in a nonhomogeneous medium. For the case with the linear drift, the solution is stationary in the long-time limit and it represents the Lévy process with a simple scaling. The solution for the drift term in the form $\lambda\,\mathrm{sgn}(x)$ possesses two different scales which correspond to the Lévy indexes μ and $\mu+1$ ($\mu<1$). The former component of the solution prevails at large distances but it diminishes with time for a given x. The fractional moments, as a function of time, are calculated. They rise with time and the rate of this growth increases with λ.

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Transport processes in physical systems can be very complex. The traditional statistical description, which relies on uniform Gaussian noises and position-independent transition probabilities, must fail for many realistic problems. A need for an alternative approach is obvious, for example, when one considers porous, irregular materials, containing impurities or entanglements which act as obstacles and dynamical traps [1]. Transport in a heterogeneous, in particular fractal, material involves position-dependent transition rates and highly complex driving forces which can be handled in a statistical manner (quenched disordered media) [2]. The Gaussian distribution not always applies. In systems which are characterized by long-range correlations and nonlocal interactions, one can expect the presence of long tails of the driving noise, i.e., one should consider the general Lévy distributions. It is so for many physical phenomena [3], in particular in biological [4,5], social [6], and epidemiological problems [7]. From the Langevin equation, driven by the homogeneous Lévy noise, follows the Fokker-Planck equation (FPE), which is fractional [8–12]. The importance of the general Lévy distribution stems from its stability: it acts as an attractor in the functional space and there are no other attractors. The physical reason behind the Lévy non-Gaussian processes traces back to the nonhomogeneous structure of the environment, in particular fractal or multifractal. However, this basis and the essential feature of the Lévy process is only rarely taken explicitly into account, and transport processes are described in terms of linear stochastic equations. The medium structure is usually reflected only in the form of the external potential or as a time dependence of the stochastic driving. In this paper, we consider a nonhomogeneous Lévy noise which leads to a position-dependent diffusion coefficient in FPE and construct the asymptotic solution of FPE.

The problem of nonhomogeneous stochastic driving can be posed in a form of the Langevin equation,

$$dx(t) = F(x)dt + \sigma(x)dL, \tag{1}$$

with the multiplicative noise which is understood in a sense of the Itô interpretation. Equation (1) can be regarded as a result of the adiabatic elimination of fast variables for nonlinear processes with additive fluctuations. In the Gaussian case, it is well suited for many such problems, e.g., the ensemble of two-level atoms in the electromagnetic field (Maxwell-Bloch equations), the parametric generation of coherent fields by incoming laser field, the Raman scattering, and the autocatalytic reactions [13]. In the present paper, we assume that the noise $L(t)$ is the uncorrelated Lévy process with the stability index μ ($0<\mu\leq 2$), the median γ, and the scale parameter K^μ. The cumulant expansion of the characteristic function, truncated at the order μ, produces the following fractional FPE [14],

$$\frac{\partial}{\partial t}p(x,t) = -\frac{\partial}{\partial x}\{[\gamma\sigma(x) + F(x)]p(x,t)\} + K^\mu \frac{\partial^\mu}{\partial |x|^\mu}[D(x)p(x,t)], \tag{2}$$

for the probability density distribution of the variable x; in the above equation $\partial^\mu/\partial|x|^\mu$ is the Riesz fractional derivative and $D(x)=|\sigma(x)|^\mu$. The initial condition is $p(x,0)=\delta(x)$. Equation (2) has been extensively studied for $D(x)=\text{const}$ [15].

On the other hand, equations in form (2) follow directly from master equation for a jumping process:

$$\dot{p}(x,t) = \int dx'[w(x|x')p(x',t) - w(x'|x)p(x,t)]. \tag{3}$$

For example, modeling the thermal activation of particles within the folded polymers leads to the FPE with the variable diffusion coefficient, which results from the polymer heterogeneity [16]. In general, $D(x)$ in FPE is variable if the transition probability $w(x|x')\neq w(|x-x'|)$. It is the case also for a coupled continuous time random walk (CTRW) which is defined in terms of a Poissonian waiting time distribution with a variable jumping rate $\nu(x)$, as well as of the Lévy distribution of jumping size $Q(x)$ [17,18]. Then $w(x|x')=\nu(x')Q(|x-x'|)$; FPE takes form (2) with $D(x)=\nu(x)$. The parameter γ can depend, in general, on the position before the jump. If it is constant, the drift term takes the form $\sim\gamma\nu(x)$, where $\gamma=\langle x\rangle_Q$. The solution of that equation for the power law $\nu(x)$ and $\gamma=0$ is still the Lévy distribution [18].

In this paper we demonstrate that the stability property holds also for some systems which contain the driving term

$F(x)$. That is by no means obvious; for example, the separable solution of the fractional Schrödinger equation in Ref. [16], which is characterized by the exponential diffusion coefficient and a periodic potential, looses its dependence on μ altogether, in the asymptotic limit. For a system with a power-law external potential, $|x|^c$, driven by the Lévy noise with the constant diffusion coefficient [19], the stochastic properties depend on c: the Lévy index in the stationary solution remains unchanged for the harmonic potential, whereas for larger powers the asymptotics is determined by c. As a result, the variance can be finite. We will solve Eq. (2) for the power-law diffusion coefficient, $D(x)=|x|^{-\theta}$ [20], and for two simplest forms of $F(x)$ which correspond to symmetric potentials.

It is convenient to handle Lévy processes by means of the Fox functions. If the solution of the FPE is to be the Lévy process, one can expect that it has the following scaling form

$$p(x,t) = Na(t)H_{2,2}^{1,1}\left[a(t)|x| \left| \begin{matrix} (a_1,A_1),(a_2,A_2) \\ (b_1,B_1),(b_2,B_2) \end{matrix} \right. \right], \quad (4)$$

where N is the normalization constant. Expression (4) is universal for any Lévy process [21]. We assume the solution of Eq. (2) in this form. To find the coefficients, we require that Eq. (2) is satisfied by Eq. (4) in the diffusion (fluid) limit of small wave numbers k. Therefore, the solution in the form (4) should coincide with the exact solution of Eq. (2) for large $|x|$.

We begin with the case $F(x)=-\lambda x$ ($\lambda>0$) which corresponds to the harmonic-oscillator potential. Moreover, we assume $\gamma=0$ and $\mu+\theta>0$. The FPE takes the form:

$$\frac{\partial}{\partial t}p(x,t) = \lambda\frac{\partial}{\partial x}[xp(x,t)] + K^\mu\frac{\partial^\mu}{\partial|x|^\mu}[|x|^{-\theta}p(x,t)], \quad (5)$$

and the Fourier transformation produces the result

$$\frac{\partial}{\partial t}\tilde{p}(k,t) = -\lambda k\frac{\partial}{\partial k}\tilde{p}(k,t) - K^\mu|k|^\mu\mathcal{F}_c[|x|^{-\theta}p(x,t)]. \quad (6)$$

To solve Eq. (6), we follow the procedure from Refs. [18,22], where also the appropriate formulas are provided. Due to the multiplication rule, the argument of the Fourier transform in the last term is the Fox function of the same order as $p(x,t)$. We apply the formula for the cosine Fourier transform and expand the results in the fractional powers of $|k|$. In order to adjust the terms on both sides of Eq. (6), we have to introduce conditions on the powers and to eliminate some of the terms by an appropriate choice of the coefficients. As a result, we determine the following coefficients of the Fox function: $a_1=1-(1-\theta)/(\mu+\theta)$, $A_1=1/(\mu+\theta)$, $b_2=1-(1-\theta)/(2+\theta)$, and $B_2=1/(2+\theta)$. Those values are—for any choice of the other parameters—a sufficient condition for Eq. (4) to represent the Lévy process in the lowest order of the k expansion:

$$\tilde{p}(k,t) \approx 1 - h_\mu a^{-\mu}|k|^\mu. \quad (7)$$

For the Fourier transform in the last term in Eq. (6), we need to keep only the term k^0. We obtain $\tilde{p}_\theta=\mathcal{F}_c[|x|^{-\theta}p(x,t)]\approx h_0a^\theta$, where a^θ results from the transformation to the scaled variable. The neglected terms on both sides of Eq. (6) are of the order $|k|^{2\mu+\theta}$. The expansion of the Fox function around zero and ∞ shows that (b_1,B_1) corresponds to the behavior of $p(x,t)$ at $x=0$, whereas (a_1,A_1) determine the asymptotics ($|x|\to\infty$). Therefore, the former ones cannot be determined in the small k approximation. We assume values which correspond to the small $|x|$ limit of the master-equation solution for CTRW [22]: $b_1=\theta$ and $B_1=1$; that process is described by Eq. (5) with $\lambda=0$. To settle a_2 and A_2, which only weakly influence $p(x,t)$ in the asymptotic limit, we require that the x dependence of the distribution [Eq. (4)] for $\lambda=0$ should coincide with the stretched-Gaussian exact solution in the limit $\mu\to 2$ [22]; then: $a_2=1/2+(1-\theta)/(2+\theta)$ and $A_2=(1+\theta)/(2+\theta)$. The coefficient h_μ follows directly from the expansion formula:

$$h_\mu = N(\mu+\theta)\Gamma(-\mu)\Gamma(1+\mu+\theta)\cos(\pi\mu/2)/$$
$$\Gamma[1/2+(\mu+\mu\theta+2)/(2+\theta)]\Gamma[-(\mu+\theta)/(2+\theta)],$$

whereas $h_0=\int_0^\infty p_\theta(ax,t)d(ax)=\lim_{s\to-1}\chi(s)=N(\mu+\theta)/(2+\theta)\Gamma[1/2-(\theta^2+\theta-2)/(2+\theta)]$, where $\chi(s)$ is the Mellin transform from the Fox function. Similarly we obtain the normalization constant: $N=\Gamma[-\theta/(2+\theta)]\Gamma[1/2+2/(2+\theta)]/2\Gamma(1+\theta)\Gamma[-\theta/(\mu+\theta)]$. Inserting $\tilde{p}$ and $\tilde{p}_\theta$ into Eq. (6) yields the equation for $a(t)$:

$$\dot{a} = \lambda a - K^\mu\frac{h_0}{\mu h\mu}a^{\mu+\theta+1}, \quad (8)$$

which can be solved by separation of the variables:

$$a(t) = \left[\frac{\lambda/c_L}{1-\exp[-\lambda(\mu+\theta)t]}\right]^{1/(\mu+\theta)}, \quad (9)$$

where $c_L=K^\mu h_0/\mu h_\mu$. For $\theta=0$, the above solution agrees with that of Ref. [10].

Expansion of the Fox function in powers of $1/|x|$ reveals the asymptotics of the Lévy process: $p(x,t)\sim a(t)^{-\mu}|x|^{-\mu-1}$ for $|x|\to\infty$. $a(t)$ approaches with time a constant which corresponds to the stationary solution of FPE. The speed of that convergence depends on θ: it is rapid for large θ, whereas negative values of θ can substantially hamper the convergence. The meaning of the parameter θ in the context of the diffusion process becomes clear when we consider the case for which the variance exists, namely, the Gaussian case $\mu=2$; it is involved in solution (4). The coefficients of the Fox function on its main diagonal are then equal and we can apply the reduction formula. The solution reads

$$p(x,t) = Na(t)H_{1,1}^{1,0}\left[a(t)|x| \left| \begin{matrix} \left(\frac{1}{2}+\frac{1-\theta}{2+\theta},\frac{1+\theta}{2+\theta}\right) \\ (\theta,1) \end{matrix} \right. \right]. \quad (10)$$

The contribution to the Barnes-Mellin integral from the residues vanish for large $|x|$ and the asymptotic form of the Fox function is stretched exponential [22,23]:

$$p(x,t) \sim a^{1+\theta}|x|^\theta\exp[-c_2(a|x|)^{2+\theta}], \quad (11)$$

where $a(t)$ is given by Eq. (9) with $\mu=2$ and $c_2=(1+\theta)^{2+\theta}/(2+\theta)^{3+\theta}$. The variance can be easily evaluated:

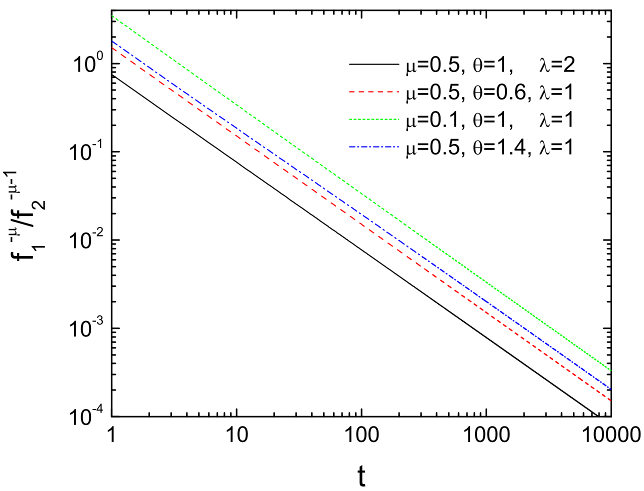

FIG. 1. (Color online) The ratio which determines the relative contribution to the solution of Eq. (13) from the terms which correspond to the Lévy indexes μ and $\mu+1$ for various parameters of the process. All curves fall like $1/t$.

$$\langle x^2\rangle = -\frac{\partial^2}{\partial k^2}\tilde{p}(0,t) = h_2 a^{-2}, \qquad (12)$$

where $h_2=\lim_{\mu\to 2} h_\mu$. If $\lambda=0$, Eq. (12) predicts the normal diffusion ($\theta=0$), subdiffusion ($\theta>0$), and superdiffusion ($\theta<0$). Otherwise, the variance converges with time to a constant. Therefore, the parameter θ governs the transport speed. In the coupled CTRW, a large θ means that the average trapping time strongly rises with the distance.

The system, which has been discussed above, is characterized by the same stability property as that of the driving noise: it is Lévy distributed with the parameter μ. Moreover, it reveals the simple scaling. One can ask whether the same properties hold for other systems driven by the Lévy noise and a symmetric potential [24]. The next case demonstrates that, even if the stability property is preserved, the index μ may change. Let us consider the drift in the form $F(x)=\lambda\,\mathrm{sgn}(x)$, which corresponds to the wedge-shaped potential. We assume $0<\mu<1$, i.e., a process of the infinite mean, and $\mu+\theta>1$. The FPE is the following

$$\frac{\partial}{\partial t}p(x,t) = \lambda\frac{\partial}{\partial x}[\mathrm{sgn}(x)p(x,t)] + K^\mu\frac{\partial^\mu}{\partial|x|^\mu}[|x|^{-\theta}p(x,t)]. \qquad (13)$$

Its cosine Fourier transform reads

$$\frac{\partial}{\partial t}\tilde{p}(k,t) = -\lambda k\frac{\partial}{\partial k}\mathcal{F}_c[|x|^{-1}p(x,t)] - K^\mu|k|^\mu\mathcal{F}_c[|x|^{-\theta}p(x,t)], \qquad (14)$$

and the factor $|x|^{-1}$, which results from the change of the sine transform to the cosine one, introduces a new scale. We take into account that double scaling by assuming the solution in the form $p(x,t)=N[p_1(x,t)+\alpha p_2(x,t)]$, where

$$p_i(x,t) = f_i(t)H^{1,1}_{2,2}\left[f_i(t)|x|\left|\begin{matrix}(a_1^{(i)},A_1^{(i)}),(a_2^{(i)},A_2^{(i)})\\(b_1^{(i)},B_1^{(i)}),(b_2^{(i)},B_2^{(i)})\end{matrix}\right.\right], \qquad (15)$$

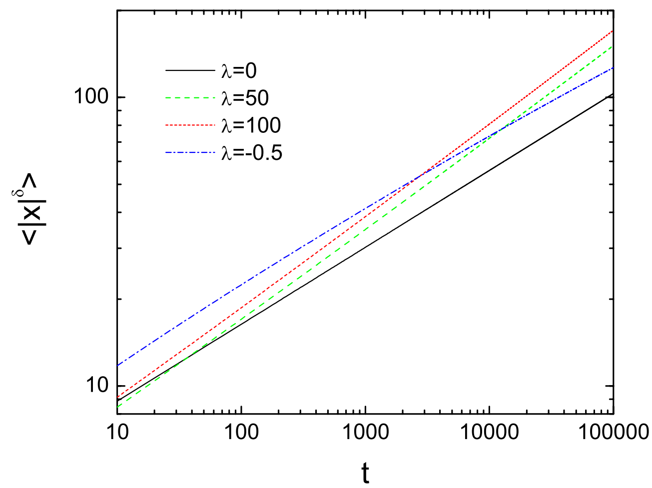

FIG. 2. (Color online) The fractional moments calculated from Eq. (17) for $\delta=0.4$, $\mu=0.5$, $\theta=1$, and a few values of λ.

and α is determined by the initial condition $p(x,0)=\delta(x)=[\delta(x)+\alpha\delta(x)]/(1+\alpha)$; we assume that $\alpha\neq 0$ if $\lambda\neq 0$. The solving method is similar to the previous case: we insert Eq. (15) into Eq. (14) and compare the terms. The essential point is to realize that the first term on right-hand side upgrades the index μ by one because the expansions of $\mathcal{F}_c[|x|^{-1}p_i(x,t)]$ are determined by the terms $|k|^{(2-a_1^{(i)})/A_1^{(i)}}$. Then $\mathcal{F}_c[|x|^{-1}p_i(x,t)]\approx \mathrm{const}-h'^{(1)}_\mu f_1^{-\mu}|k|^{\mu+1}$, where terms of the order $\mu+2$ have been neglected. We put $a_i^{(1)}=a_i$, $A_i^{(1)}=A_i$, $b_i^{(1)}=b_i$, and $B_i^{(1)}=B_i$. The coefficients for p_2 are the same except $\mu\to\mu+1$. By comparing the terms of order μ and $\mu+1$, we obtain a set of two differential equations

$$\dot{\xi}_1 = K^\mu\left[\frac{h_0^{(1)}}{h_\mu^{(1)}}\xi_1^{-\theta/\mu} + \alpha\frac{h_0^{(2)}}{h_\mu^{(1)}}\xi_2^{-\theta/(\mu+1)}\right],$$

$$\dot{\xi}_2 = (\mu+1)\frac{\lambda h'^{(1)}_\mu}{\alpha h_\mu^{(2)}}\xi_1, \qquad (16)$$

where $\xi_1=f_1^{-\mu}$, $\xi_2=f_2^{-\mu-1}$, $h_\mu^{(1)}=h_\mu$, $h_\mu^{(2)}=h_{\mu+1}$, $h_0^{(1)}=h_0$, $h_0^{(2)}=h_0(\mu\to\mu+1)$, and

$$h'^{(1)}_\mu = -N(\mu+\theta)\Gamma(-\mu-1)\Gamma(2+\mu)\sin(\pi\mu/2)/$$
$$\Gamma[1/2+(\mu+\mu\theta+2)/(2+\theta)]\Gamma[-(\mu+\theta)/(2+\theta)].$$

Let us assume $\lambda>0$. The asymptotic form of $p(x,t)$ involves contributions from both μ and $\mu+1$, $p(x,t)=c_1f_1^{-\mu}|x|^{-\mu-1}+c_2f_2^{-\mu-1}|x|^{-\mu-2}+o(|x|^{-2\mu-\theta-1})$, where c_1 and c_2 are constants. Therefore, the long tails which correspond to index μ prevail at large distances and the mean value is infinite. However, in the limit of long time the relative contribution to $p(x,t)$ from p_1 diminishes for a given x since $f_2(t)$ falls faster than $f_1(t)$. To demonstrate that, we need to estimate the ratio $\xi_1/\xi_2=f_1^{-\mu}/f_2^{-\mu-1}$. That quantity is presented in Fig. 1. Its time dependence can be very well reproduced by the function $1/t$ and this pattern is generic for all values of λ, μ, and θ. Consequently, the contribution from p_1, which originates from the Lévy process with the order parameter μ, gradually fades away. Instead, the term corresponding to $\mu+1$ dominates the distribution at large time and p_1 enters the asymptotic expression only with a small weight.

The transport in superdiffusive systems is used to be characterized by fractional moments of the order δ, $\langle|x|^\delta\rangle$, where $\delta<\mu$, since all higher moments, in particular the variance, are divergent. That moment is easy to evaluate as the Mellin transform χ_i from the functions $p_i(x,t)$:

$$\langle|x|^\delta\rangle = 2\int_0^\infty x^\delta p(x,t)dx$$
$$= 2N[f_1^{-\delta}\chi_1(-\delta-1) + \alpha f_2^{-\delta}\chi_2(-\delta-1)]$$
$$\sim\Gamma\left(-\frac{\theta+\delta}{\mu+\theta}\right)f_1^{-\delta} + \alpha\Gamma\left(-\frac{\theta+\delta}{\mu+\theta+1}\right)f_2^{-\delta}. \quad (17)$$

Figure 2 presents the fractional moments for a few values of the parameter λ. The dependence on time is algebraic and the power rises with λ; for λ=0 the moment is $\sim t^{\delta/(\mu+\theta)}$.

Finally, let us mention the case of the attractive potential, $\lambda<0$. We require that also $\alpha<0$, in order to avoid shrinking of p_2 with time [c.f. second equation in Eq. (16)], which is unphysical. The negative α, in turn, results in negative $p(x,t)$ for large time since p_1 falls faster with time than p_2. Therefore, the above solution of Eq. (13) is correct for $\lambda<0$ only if time is not very large. The fractional moment rises slower than for λ=0 and also weaker than algebraically, which is presented in Fig. 2.

In summary, we have studied the stochastic systems in which the nonhomogeneous structure of the medium is reflected not only by an external potential but also directly by the random force in the form of the Lévy distribution. Such systems are described by the fractional FPE with the variable diffusion coefficient; they have been solved in the limit of small wave numbers. The system can have the same stability properties as the driving noise; it is the case for the linear drift. However, the other drift we have considered, $\sim\text{sgn}(x)$, requires the additional Lévy process and then the system has the double scaling. That second Lévy distribution is characterized by larger order parameter and its weight rises with time.

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